ON INHOMOGENEOUS STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER EQUATION

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Abstract. In this paper we consider inhomogeneous Strichartz estimates in the mixed norm spaces which are given by taking the temporal integration before the spatial integration. We obtain some new estimates, and discuss about the necessary conditions.

1. Introduction

To begin with, let us consider the Cauchy problem

$$\begin{cases} iu_t + \Delta u = F(x,t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x). \end{cases}$$

By Duhamel's principle we have the solution

$$u(x,t) = e^{it\Delta} f(x) - i \int_0^t e^{i(t-s)\Delta} F(s) ds.$$

Here $e^{it\Delta}$ is the free propagator which is given by

$$e^{it\Delta} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)} \widehat{f}(\xi) d\xi.$$

The estimates for the solution in terms of f and F play important roles in the study of nonlinear Schrödinger equations (cf. [4, 22]). The control of solution u actually consists of two parts, homogeneous ($F = 0$) and inhomogeneous ($f = 0$) part.

It is well known that the homogeneous Strichartz estimate

$$\big\| e^{it\Delta} f \big\|_{L_t^q L_x^r} \le C \|f\|_2 \tag{1.1}$$

holds if and only if $2/q = n(1/2 - 1/r)$, $q \ge 2$ and $(q,r,n) \ne (2,\infty,2)$ (see [11, 13] and references therein). But determining the optimal range of (q,r)

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and $(\widetilde{q}',\widetilde{r}')$ for which the inhomogeneous Strichartz estimate

$$\Big\| \int_0^t e^{i(t-s)\Delta} F(s)ds \Big\|_{L_t^q L_x^r} \le C\|F\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}} \tag{1.2}$$

holds is not completed yet. By duality the homogeneous estimates imply some inhomogeneous estimates but it was observed that the estimate (1.2) is valid on a wider range than what is given by admissible pairs (q,r), $(\widetilde{q}',\widetilde{r}')$ for the homogeneous estimates (1.1) (see [6], [12]). Foschi and Vilela in their independent works ([10],[24]) obtained the currently best known range of (q,r) and $(\widetilde{q}',\widetilde{r}')$ for which (1.2) holds. However, there still remain some gaps between their range and the known necessary conditions. Also, see [18] for a new necessary condition and some weak endpoint estimates.

1.1. **Time-space estimates.** We now consider estimates in different mixed norms which are given by taking time integration before spatial integration. We call (1.1) and (1.2) *space-time estimate*, and by *time-space estimate* we mean the estimate given in $L_x^r L_t^q$ norms; e.g. (1.3), (1.4). There are estimates similar to the space-time estimates (1.1). More generally, the estimates

$$\|e^{it\Delta} f\|_{L_x^r L_t^q} \le C\|f\|_{\dot{H}^s}, \quad s = n/2 - 2/q - n/r, \tag{1.3}$$

have been of interest. Here $\dot{H}^s$ denotes the homogeneous Sobolev space of order s. Even though (1.1) and (1.3) have the same scaling, they are of different natures. Especially, for time-space estimate Galilean invariance is no longer valid in general. The condition $1/q + (n+1)/r \le n/2$ is necessary for (1.3) even with frequency localized initial datum f as it is easily seen by using Knapp's example. It is currently conjectured that (1.3) holds whenever $1/q + (n+1)/r \le n/2$, $2 \le q < \infty$. When $n = 1$, it is known to be true [14]. In higher dimensions (1.3) is verified for q, r satisfying $1/q + (n+1)/r \le n/2$, additionally $r > 16/5$ when $n = 2$, and $r > 2(n+3)/(n+1)$ when $n \ge 3$ ([16]). The estimate (1.3) is closely related to the maximal Schrödinger estimate which has been studied to obtain almost everywhere convergence to initial data. See [3, 8, 20, 23, 14, 16, 19] and references therein for further discussions and related issues. Also see [15, 1] for recent results.

In this paper we aim to look for the optimal range of $(\widetilde{r}', r)$ for which the time-space inhomogeneous Strichartz estimate

$$\Big\| \int_0^t e^{i(t-s)\Delta} F(s)ds \Big\|_{L_x^r L_t^q} \le C\|F\|_{L_x^{\widetilde{r}'} L_t^{\widetilde{q}'}} \tag{1.4}$$

holds for some $q, \widetilde{q}'$. Obviously, this is weaker than (1.2) if $q \le r$ and $\widetilde{q}' \ge \widetilde{r}'$ since one can get (1.4) from (1.2) via Minkowski's inequality. However, as it turns out, the range for (1.4) is quite different from that of (1.2). The currently known range of $(1/\widetilde{r}', 1/r)$ for which (1.2) is valid for some $q, \widetilde{q}'$ is contained in the closed pentagon with vertices $(1/2, 1/2)$, C', S', S, C (see Figure 1) and it is known that (1.2) fails unless $(1/\widetilde{r}', 1/r)$ is contained in the closed pentagon with vertices $(1/2, 1/2)$, C', R', R, C. We will show that (1.4) is possible only if $(1/\widetilde{r}', 1/r)$ is contained in the closed trapezoid B, R, R', B' from which the points R, R' are removed. In [9] it was shown that if $1 \le \widetilde{r}' \le 2 \le r \le \infty$ and $|1/r + 1/\widetilde{r}' - 1| < 1/n$, there are $q, \widetilde{q}'$ which allow the time delayed estimates in time-space norm. But in contrast to the space-time estimate (1.2) the above discussion shows that mere existence of such $q, \widetilde{q}'$ for time delayed estimate is not enough to obtain (1.4) and accurate information on the possible range of $q, \widetilde{q}'$ is important.

To show (1.4) we work on Fourier transform side by making use of the fact that the Duhamel part is similar to multiplier of negative order (see [7, 17]). This allows us to take advantage of localization in Fourier transform side which plays important roles in our argument. We believe that this method is more flexible than the conventional argument which heavily relies on the dispersive estimate.

Necessary conditions. We now discuss the conditions on (q, r) and $(\widetilde{q}', \widetilde{r}')$ which are necessary for (1.4). By scaling the condition

$$\frac{1}{\widetilde{q}'} - \frac{1}{q} + \frac{n}{2}\left(\frac{1}{\widetilde{r}'} - \frac{1}{r}\right) = 1 \tag{1.5}$$

should be satisfied. Using the examples in [10, 24], we see that the conditions which are needed for (1.2) are also necessary for (1.4):

$$\widetilde{r}' < 2 < r, \quad \frac{1}{\widetilde{r}'} - \frac{1}{r} \le \frac{2}{n}, \quad 1 - \frac{1}{n} \le \frac{1}{\widetilde{r}'} + \frac{1}{r} \le 1 + \frac{1}{n}, \tag{1.6}$$

$$\widetilde{q}' \le q, \quad \frac{1}{q} < n\left(\frac{1}{2} - \frac{1}{r}\right), \quad \frac{1}{\widetilde{q}'} > 1 - n\left(\frac{1}{\widetilde{r}'} - \frac{1}{2}\right). \tag{1.7}$$

By considering additional test functions, we get the following conditions which will be shown later (see Section 4):

$$\frac{1}{\widetilde{q}'} - \frac{1}{q} + (n+1)\Big(\frac{1}{\widetilde{r}'} - \frac{1}{r}\Big) \geq 2, \tag{1.8}$$

$$\frac{1}{\widetilde{q}'} - \frac{1}{q} \geq \frac{2n}{r} - n + 1, \quad \frac{1}{\widetilde{q}'} - \frac{1}{q} \geq n + 1 - \frac{2n}{\widetilde{r}'}. \tag{1.9}$$

To facilitate the statement of our results, for $n \geq 3$, let us define points B, C, P, Q, R, and S which are contained in $[\frac{1}{2}, 1] \times [0, \frac{1}{2}]$ by setting

$$B = \Big(\frac{n+3}{2(n+2)}, \frac{n-1}{2(n+2)}\Big), \ \ C = \Big(\frac{1}{2}, \frac{n-2}{2n}\Big), \ \ P = \Big(\frac{n+2}{2(n+1)}, \frac{n^2}{2(n+1)(n+2)}\Big),$$

$$Q = \Big(\frac{n+2}{2(n+1)}, \frac{n-2}{2(n+1)}\Big), \ \ R = \Big(\frac{n+1}{2n}, \frac{n-3}{2n}\Big), \ \ S = \Big(\frac{n}{2(n-1)}, \frac{(n-2)^2}{2n(n-1)}\Big),$$

and we also define the dual points B', C', P', Q', R', S' by setting $X' = (1-b, 1-a)$ when $X = (a,b)$. (See Figure 1.) Let $\mathcal{N}(n)$ be the closed trapezoid with vertices B, B', R, R' from which the points R, R' are removed. Being combined with (1.5), (1.8) gives

$$\frac{1}{\widetilde{r}'} - \frac{1}{r} \geq \frac{2}{n+2}, \tag{1.10}$$

and the first and second conditions in (1.9) give

$$n\big(1 - \frac{1}{2\widetilde{r}'}\big) \geq \frac{3n}{2r}, \quad \frac{3n}{2\widetilde{r}'} \geq n\big(1 - \frac{1}{2r}\big), \tag{1.11}$$

respectively. Also, by (1.5) and (1.7), we see that $(1/\widetilde{r}', 1/r) \neq R, R'$. Hence, from this, (1.6), (1.10) and (1.11), it follows that (1.4) holds only if $(1/\widetilde{r}', 1/r) \in \mathcal{N}(n)$.

Sufficiency part. We will show a stronger estimate

$$\Big\| \int_{-\infty}^{t} e^{i(t-s)\Delta} F(s) ds \Big\|_{L_x^r L_t^q} \leq C \|F\|_{L_x^{\widetilde{r}'} L_t^{\widetilde{q}'}}, \tag{1.12}$$

which implies (1.4) and $\| \int_{-\infty}^{\infty} e^{i(t-s)\Delta} F(s) ds\|_{L_x^r L_t^q} \leq C\|F\|_{L_x^{\widetilde{r}'} L_t^{\widetilde{q}'}}$. As mentioned above, if $q \leq r$ and $\widetilde{q}' \geq \widetilde{r}'$, from the known range of the space-time estimate ([10, 23]), one can get (1.12) for $(1/\widetilde{r}', 1/r)$ contained in the closed hexagon $\mathcal{H}$ with vertices P, Q, S, P', Q', S' from which the line segments $[P,Q], [P',Q']$ and the points S, S' are removed [1]. We extend the range further to include the triangular region ΔQRS and $\Delta Q'R'S'$. It should be noted that

[1] In fact, when $q \leq r$ and $\widetilde{q}' \geq \widetilde{r}'$, (1.5) and (1.7) are satisfied if $(1/\widetilde{r}', 1/r)$ is contained in $\mathcal{H}$. So we can use the known space-time estimate.

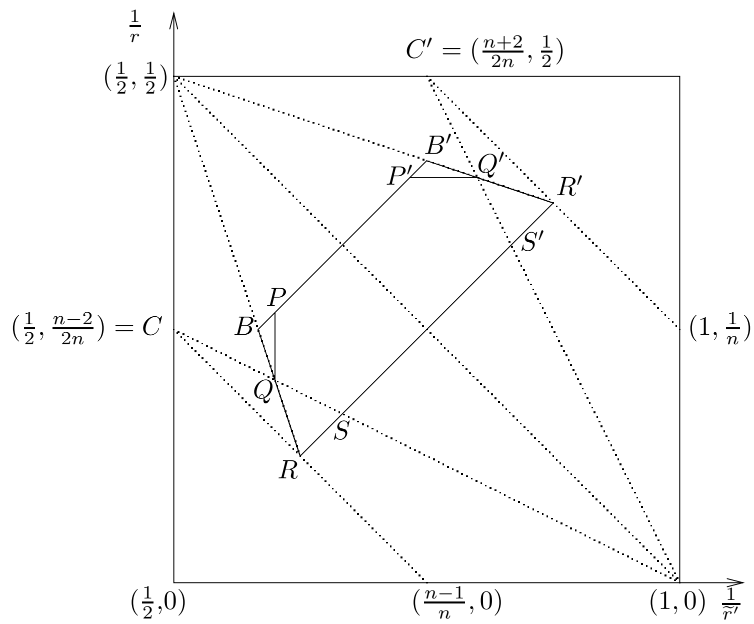

FIGURE 1. The points B, C, P, Q, R, S and the dual points B', C', P', Q', R', S' when $n \geq 3$.

no inhomogeneous space-time estimate (1.2) is known for $(1/\widetilde{r}', 1/r)$ which is contained in the interior of ΔQRS and $\Delta Q'R'S'$.

Theorem 1.1. *Let $n \geq 3$ and $\mathcal{S}(n)$ be the open hexagon with vertices P, Q, R, P', Q', R' to which the line segments (P, P') and (R, R') are added. If $(1/\widetilde{r}', 1/r) \in \mathcal{S}(n)$, then* (1.12) *holds for some $q, \widetilde{q}'$.*

For $(1/\widetilde{r}', 1/r)$ contained in the region $\Delta QRS \setminus [Q, R]$, the estimate (1.12) is available if $(\widetilde{q}', q)$ satisfies (1.5), (1.7) and additionally $\frac{1}{q} < n(\frac{1}{\widetilde{r}'} - \frac{1}{2})$, $\frac{1}{\widetilde{q}'} > 1 - n(\frac{1}{2} - \frac{1}{r})$. Being combined with (1.5), these additional conditions are due to the third inequality of (3.2) and its dual one. By duality the same holds for $(1/\widetilde{r}', 1/r)$ which is contained in the region $\Delta Q'R'S' \setminus [Q', R']$. Making use of the known time-space homogeneous estimate (1.3) together with the argument of this paper, it is possible to obtain further estimates which extend the range of $q, \widetilde{q}'$ but it does not extend the range of $(\widetilde{r}', r)$.

When $n = 2$, (1.12) holds if $(1/\widetilde{r}', 1/r)$ is contained in the open pentagon with vertices P, Q, $(1, 0)$, Q', P' to which the line segment (P, P') is added but it is not new. This just follows from the known range of the space-time estimate ([10, 24]). When $n = 1$, it is possible to obtain the full range except some

endpoint estimates. In fact, from the necessary conditions, (1.4) is possible only if $(1/\widetilde{r}', 1/r)$ is contained in the closed triangle Δ with vertices $(\frac{2}{3}, 0)$, $(1, 0)$, $(1, \frac{1}{3})$.

Theorem 1.2. *Let* $n = 1$. *Then* (1.12) *holds for some* q, $\widetilde{q}'$ *provided that* $(1/\widetilde{r}', 1/r)$ *is contained in* $\Delta \setminus \big([(\frac{2}{3}, 0), (1, 0)] \cup [(1, \frac{1}{3}), (1, 0)]\big)$. *In fact,* (1.12) *holds if* q, $\widetilde{q}'$ *satisfies* $1 < \widetilde{q}' < 2 < q < \infty$ *and* $1/\widetilde{r}' - 1/r + 1/2\widetilde{q}' - 1/2q \geq 1$.

The rest of this paper is organized as follows: In Section 2 we obtain some frequency localized estimates which will be used in later sections. Then, using these estimates and a summation method, we prove Theorem 1.1 and 1.2 in Section 3. Nextly, we show the necessary conditions (1.8) and (1.9) in Section 4.

Throughout this paper, the letter C stands for a constant which is possibly different at each occurrence. In addition to the symbol $\widehat{\ }$, we use $\mathcal{F}(\cdot)$ to denote the Fourier transform, and $\mathcal{F}^{-1}(\cdot)$ to denote the inverse Fourier transform. Finally, we denote by χ_E the characteristic function of a set E.

2. Preliminaries

In this section we prove several preliminary estimates which will be used for the proof of Theorem 1.1, which is to be shown in Section 3.

Let us define the operator T_δ for dyadic numbers $\delta \in 2^{\mathbb{Z}} := \{2^z : z \in \mathbb{Z}\}$ by

$$T_\delta F = \int \delta\phi(\delta(t-s))e^{i(t-s)\Delta}F(s)ds \tag{2.1}$$

where ϕ is a smooth function supported in $(1/2, 2)$ such that $\sum_{k=-\infty}^{\infty}\phi(2^k t) = 1$, $t > 0$. Then we may write

$$TF := \int_{-\infty}^{t} e^{i(t-s)\Delta}F(s)ds = \sum_{\delta\in 2^{\mathbb{Z}}} \delta^{-1}T_\delta F. \tag{2.2}$$

By direct computation it is easy to see that

$$\widehat{T_\delta F}(\xi,\tau) = \widehat{\phi}(\frac{\tau + |\xi|^2}{\delta})\widehat{F}(\xi,\tau). \tag{2.3}$$

By this dyadic decomposition in time, the boundedness problem for T is essentially reduced to obtaining suitable bounds for T_δ in terms of δ. From this one may view the operator $F \to \int_{-\infty}^{t} e^{i(t-s)\Delta}F(s)ds$ as the multiplier operator of negative order 1 which is associated to the paraboloid.

Proposition 2.1. *Let $n \ge 2$. Suppose that Fourier transform of F is supported in $\{(\xi,\tau)\in\mathbb{R}^n\times\mathbb{R} : 1/2 \le |\xi| \le 2\}$. Then we have*

$$\|T_\delta F\|_{L^r_x L^2_t} \le C\delta^{-\frac{n-1}{2}+\frac{n}{\widetilde{r}'}}\|F\|_{L^{\widetilde{r}'}_x L^2_t}$$

for $r, \widetilde{r}'$ satisfying $1 \le \widetilde{r}' \le 2$ and $(n+1)/r \le (n-1)(1-1/\widetilde{r}')$.

Proof. In view of interpolation it is enough to consider the cases $(\widetilde{r}', r) = (2, \frac{2(n+1)}{n-1})$ and $(1,\infty)$. This actually gives the estimates along the line $(n+1)/r = (n-1)(1-1/\widetilde{r}')$. The other estimates follow from Bernstein's inequality because the spatial Fourier transform of F is compactly supported.

The case $(\widetilde{r}', r) = (2, \frac{2(n+1)}{n-1})$. By duality it is enough to show that

$$\|T_\delta F\|_{L^2_x L^2_t} \le C\delta^{\frac12}\|F\|_{L_x^{\frac{2n+2}{n+3}} L^2_t}.$$

Since we are assuming that $\widehat{F}(\cdot,\tau)$ is supported in $\{|\xi| \sim 1\}$, by (2.3) and Plancherel's theorem we have

$$\|T_\delta F\|^2_{L^2_x L^2_t} \le C \iint_{1/2\le|\xi|\le 2} \left|\widehat{\phi}\left(\frac{\tau+|\xi|^2}{\delta}\right)\widehat{F}(\xi,\tau)\right|^2 d\xi d\tau.$$

So we are reduced to showing that

$$\iint_{1/2\le|\xi|\le 2} \left|\widehat{\phi}\left(\frac{\tau+|\xi|^2}{\delta}\right)\widehat{F}(\xi,\tau)\right|^2 d\xi d\tau \le C\delta\|F\|^2_{L_x^{\frac{2n+2}{n+3}} L^2_t}. \tag{2.4}$$

Then the left hand side equals to

$$\iint_{\frac12}^2 \left|\widehat{\phi}\left(\frac{\tau+r^2}{\delta}\right)\right|^2 \int_{S^{n-1}} |\mathcal{F}_x\mathcal{F}_t F(r\theta,\tau)|^2 d\theta\, r^{n-1} dr d\tau.$$

Using Tomas-Stein theorem [21] (L^2-restriction estimate to the sphere rS^{n-1}, $r\sim 1$), we see that

$$\int_{S^{n-1}} |\mathcal{F}_x\mathcal{F}_t F(r\theta,\tau)|^2 d\theta \le C\|\mathcal{F}_t F(\cdot,\tau)\|^2_{L_x^{\frac{2n+2}{n+3}}}.$$

Taking integration in r, it follows that

$$\iint_{1/2\le|\xi|\le 2} \left|\widehat{\phi}\left(\frac{\tau+|\xi|^2}{\delta}\right)\widehat{F}(\xi,\tau)\right|^2 d\xi d\tau \le C\delta\|\mathcal{F}_t F(\tau)\|^2_{L^2_\tau L_x^{\frac{2n+2}{n+3}}}.$$

By Minkowski's inequality and Plancherel's theorem, we get (2.4).

The case $(\widetilde{r}', r) = (1,\infty)$. Note that $T_\delta F$ can be written as

$$T_\delta F(x,t) = \iint K_\delta(x-y, t-s)F(y,s)dyds,$$

where

$$K_\delta(y,s) = \delta\phi(\delta s)\iint e^{i(-sr^2+ry\cdot\theta)}\psi(r)d\theta dr \tag{2.5}$$
$$= \iiint \widehat{\phi}(\frac{\tau+r^2}{\delta})e^{i(s\tau+ry\cdot\theta)}\psi(r)d\theta dr d\tau$$

and $\psi \in C_0^\infty(1/2,2)$. Since $|K_\delta(y,s)| \le C|\delta\phi(\delta s)|$, by Young's inequality, we have $\|T_\delta F\|_{L^\infty_x L^2_t} \le C\|F\|_{L^1_x L^2_t}$. So we may assume that $\delta \lesssim 1$.

By the choice of ϕ, $K_\delta \neq 0$ for $s \sim \delta^{-1}$. Hence by non-stationary phase method, we see that $|K_\delta(y,s)| \le C\delta^M$ if $|y| \le \delta^{-1}/100$ and $|K_\delta(y,s)| \le C(1+|y|)^{-N}$ if $|y| \ge 100\delta^{-1}$. Let us set $\chi_\delta(y) = \chi_{\{\delta^{-1}/100 \le |y| \le 100\delta^{-1}\}}$, $\widetilde{K}_\delta(y,s) = K_\delta(y,s)\chi_\delta(y)$, and

$$\widetilde{T}_\delta F(x,t) = \iint \widetilde{K}_\delta(x-y,t-s)F(y,s)dyds.$$

Then it is enough to show that

$$\|\widetilde{T}_\delta F\|_{L^\infty_x L^2_t} \le C\delta^{\frac{n+1}{2}}\|F\|_{L^1_x L^2_t}. \tag{2.6}$$

Form (2.5), it follows that

$$\mathcal{F}_t(\widetilde{T}_\delta F)(x,\tau)$$
$$= \int \mathcal{F}_t F(y,\tau)\int \widehat{\phi}(\frac{\tau+r^2}{\delta})\chi_\delta(x-y)\psi(r)\left(\int_{S^{n-1}} e^{ir(x-y)\cdot\theta}d\theta\right)drdy.$$

Hence by plancherel's theorem we see that $\|\widetilde{T}_\delta F(x,\cdot)\|^2_{L^2_t}$ is bounded by

$$\int\left[\iint |\mathcal{F}_t F(y,\tau)|\int\left|\widehat{\phi}(\frac{\tau+r^2}{\delta})\chi_\delta(x-y)\psi(r)\int_{S^{n-1}} e^{ir(x-y)\cdot\theta}d\theta\right|drdy\right]^2 d\tau.$$

By using the fact that $\int e^{irx\cdot\theta}d\theta = O(|x|^{-\frac{n-1}{2}})$ for large $|x|$, and taking integration in r,

$$\|\widetilde{T}_\delta F(x,\cdot)\|^2_{L^2_t} \le C\delta^{n+1}\int\left(\int |\mathcal{F}_t F(y,\tau)|dy\right)^2 d\tau.$$

By Minkowski's inequality and Plancherel's theorem,

$$\|\mathcal{F}_t F\|_{L^2_\tau L^1_x} \le \|\mathcal{F}_t F\|_{L^1_x L^2_\tau} = \|F\|_{L^1_x L^2_t}.$$

Hence we get (2.6). $\square$

Throughout this paper we will use the following summation lemma several times which is due to Bourgain [2] (Also see [5] for a generalization.) The lemma is a version of Lemma 2.3 in [17] for Banach-valued functions. (For a proof we refer the reader to [17].)

Lemma 2.2. *Let $\varepsilon_1, \varepsilon_2 > 0$. Let $1 \le q \le \infty$ and $1 \le r_1, r_2 < \infty$. Suppose that $f_1(y,z), f_2(y,z), ..., f_j(y,z), ...$ be functions defined on $\mathbb{R}^l \times \mathbb{R}^m$ which satisfies that $\|f_j\|_{L_y^{r_1} L_z^q} \le M_1 2^{\varepsilon_1 j}$ and $\|f_j\|_{L_y^{r_2} L_z^q} \le M_2 2^{-\varepsilon_2 j}$ for $1 \le r_1, r_2 < \infty$. Then*

$$\|\sum f_j\|_{L_y^{r,\infty} L_z^q} \le C M_1^\theta M_2^{1-\theta},$$

where $\theta = \varepsilon_2/(\varepsilon_1+\varepsilon_2)$ and $1/r = \theta/r_1 + (1-\theta)/r_2$. Here we denote by $L_y^{r,\infty}$ the weak L^r space.

Using this lemma, we remove the assumption that the spatial Fourier transform of F is supported in $\{|\xi| \sim 1\}$.

Proposition 2.3. *Let $n \ge 3$. Suppose that the spatial Fourier transform of F is supported in $\{(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R} : |\xi| \le 2\}$. Then we have*

$$\|T_\delta F\|_{L_x^r L_t^2} \le C \delta^{-\frac{n-1}{2} + \frac{n}{\widetilde{r}'}} \|F\|_{L_x^{\widetilde{r}'} L_t^2}$$

for $r, \widetilde{r}'$ satisfying $n/(n-1) < \widetilde{r}' \le 2$ and $1/r + 1/\widetilde{r}' \le (n-1)/n$.

Proof. Since we are assuming that F is supported in $\{(\xi,\tau) : |\xi| \le 2\}$, we may break T_δ so that

$$T_\delta = \sum_{j \ge -1} T_\delta^j,$$

where T_δ^j is given by

$$\widehat{T_\delta^j F}(\xi,\tau) = \widehat{\phi}\Big(\frac{\tau + |\xi|^2}{\delta}\Big) \phi(2^j|\xi|) \widehat{F}(\xi,\tau).$$

From rescaling we have

$$T_\delta^j F(x,t) = T_{2^{2j}\delta} F_j(2^{-j}x, 2^{-2j}t),$$

where $F_j = \phi(|D|) F(2^j \cdot, 2^{2j} \cdot)$. Thus, by Proposition 2.1 we see that

$$\|T_\delta^j F\|_{L_x^r L_t^2} \le C \delta^{-\frac{n-1}{2} + \frac{n}{\widetilde{r}'}} 2^{jn(\frac{1}{r} + \frac{1}{\widetilde{r}'} - \frac{n-1}{n})} \|F\|_{L_x^{\widetilde{r}'} L_t^2} \tag{2.7}$$

for $r, \widetilde{r}'$ satisfying $1 \le \widetilde{r}' \le 2$ and $(n+1)/r \le (n-1)(1 - 1/\widetilde{r}')$. If $1/r + 1/\widetilde{r}' < (n-1)/n$, we can sum to get the desired estimate. To the estimates for the endpoint cases $1/r + 1/\widetilde{r}' = (n-1)/n$, we use Lemma 2.2.

Fix $\widetilde{r}', r$ such that $1/r + 1/\widetilde{r}' = (n-1)/n$ and $n/(n-1) < \widetilde{r}' < 2$. We now choose r_1, r_2, so that $(n+1)/r_i \le (n-1)(1 - 1/\widetilde{r}')$, $i = 1, 2$, and

$$\frac{1}{r_2} + \frac{1}{\widetilde{r}'} < \frac{n-1}{n} < \frac{1}{r_1} + \frac{1}{\widetilde{r}'}.$$

Note that $(1/\widetilde{r}', 1/r)$ is on the open segment joining $(1/\widetilde{r}', 1/r_1)$ and $(1/\widetilde{r}', 1/r_2)$. From (2.7) we see

$$\|T_\delta^j F\|_{L_x^{r_i} L_t^2} \le C\delta^{-\frac{n-1}{2}+\frac{n}{\widetilde{r}'}} 2^{jn(\frac{1}{r_i}+\frac{1}{\widetilde{r}'}-\frac{n-1}{n})} \|F\|_{L_x^{\widetilde{r}'} L_t^2}$$

for $i = 1, 2$. We now can apply Lemma 2.2 with $\varepsilon_i = n|\frac{1}{r_i} + \frac{1}{\widetilde{r}'} - \frac{n-1}{n}|$. So, we get

$$\|T_\delta^j F\|_{L_x^{r,\infty} L_t^2} \le C\delta^{-\frac{n-1}{2}+\frac{n}{\widetilde{r}'}} \|F\|_{L_x^{\widetilde{r}'} L_t^2}.$$

This weak type estimate for $1/r + 1/\widetilde{r}' = (n-1)/n$ and $n/(n-1) < \widetilde{r}' < 2$ can be strengthened to strong type by real interpolation. Lastly, the estimate for $(1/\widetilde{r}', 1/r) = (\frac{1}{2}, \frac{n-2}{2n})$ can be obtained directly from

$$\|T_\delta F\|_{L_t^2 L_x^{\frac{2n}{n-2}}} \le C\delta^{\frac{1}{2}} \|F\|_{L_t^2 L_x^2} \tag{2.8}$$

via Minkowski's inequality. This also follows from the endpoint space-time homogeneous estimate. Indeed, by Hölder's inequality we see

$$|T_\delta F(x,t)| \le C\delta^{\frac{1}{2}} \big\| e^{i(t-s)\Delta} F(s) \big\|_{L_s^2},$$

and so

$$\|T_\delta F\|_{L_t^2 L_x^{\frac{2n}{n-2}}} \le C\delta^{\frac{1}{2}} \big\| e^{i(t-s)\Delta} F(s) \big\|_{L_s^2 L_t^2 L_x^{\frac{2n}{n-2}}}$$

by Minkowski's inequality. By applying (1.1) with $(q, r) = (2, 2n/(n-2))$, we get (2.8). $\square$

3. Sufficiency part: Proofs of Theorem 1.1 and 1.2

In this section we will prove Theorem 1.1 and 1.2. We may assume that the space time Fourier transform of F is supported in the set $\{(\xi, \tau) : |\xi| \le 2,\ |\tau| \le 2\}$ since this additional assumption can be simply removed by rescaling together with the condition (1.5).

Proof of Theorem 1.1. Since we already have the estimates in the hexagon $\mathcal{H}$, to show (1.12) it suffices to show the estimates when $(1/\widetilde{r}', 1/r) \in \big(\Delta QRS \cup \Delta Q'R'S'\big) \setminus \big([Q,R] \cup [Q',R']\big)$. By duality and complex interpolation, it is enough to show the case where $(1/\widetilde{r}', 1/r) \in \Delta QRS \setminus ([Q,R] \cup [Q,S])$.

Let $\Omega = \Omega(n)$ denote the closed triangle with vertices C, $(\frac{n-1}{n}, 0)$, $(1, 0)$ from which the point $(\frac{n-1}{n}, 0)$ is removed. The proof is then based on the following estimate:

For $(1/\widetilde{r}', 1/r) \in \Omega$,

$$\|T_\delta F\|_{L_x^r L_t^q} \le C\delta^{\frac{1}{\widetilde{q}'}-\frac{1}{q}+\frac{n}{2}(\frac{1}{\widetilde{r}'}-\frac{1}{r})} \|F\|_{L_x^{\widetilde{r}'} L_t^{\widetilde{q}'}} \tag{3.1}$$

holds provided that

$$-\frac{n}{2}(\frac{1}{r}+\frac{1}{\widetilde{r}'}-1) \le \frac{1}{q} \le \frac{1}{\widetilde{q}'} \le 1+\frac{n}{2}(\frac{1}{r}+\frac{1}{\widetilde{r}'}-1).$$

This can be shown by interpolating the case $(1/\widetilde{r}', 1/r) = (1, 0)$ and the case in which $(1/\widetilde{r}', 1/r)$ is on the line segment joining C and $(\frac{n-1}{n}, 0)$. Since Proposition 2.3 already gives the estimates on the line segment, we need only to show that

$$\|T_\delta F\|_{L^\infty_x L^q_t} \le C\delta^{\frac{1}{\widetilde{q}'}-\frac{1}{q}+\frac{n}{2}} \|F\|_{L^1_x L^{\widetilde{q}'}_t}$$

for $1 \le \widetilde{q}' \le q \le \infty$. By Minkowski's inequality, it is enough to show that

$$\|T_\delta F\|_{L^q_t L^\infty_x} \le \delta^{\frac{1}{\widetilde{q}'}-\frac{1}{q}+\frac{n}{2}} \|F\|_{L^{\widetilde{q}'}_t L^1_x}.$$

Using the fact that $\left\|e^{i(t-s)\Delta} g\right\|_{L^\infty_x} \le C|t-s|^{-n/2}\|g\|_{L^1_x}$ (dispersive estimate), this follows from (2.1) and Young's inequality.

Now we fix $\widetilde{r}', r$ such that $(1/\widetilde{r}', 1/r) \in \Delta QRS \setminus ([Q,S] \cup [Q,R])$. We claim that there is $(1/\widetilde{q}', 1/q) \in [0,1] \times [0,1]$ which satisfies (1.5) and

$$-\frac{n}{2}(\frac{1}{r}+\frac{1}{\widetilde{r}'}-1) < \frac{1}{q} \le \frac{1}{\widetilde{q}'} < 1+\frac{n}{2}(\frac{1}{r}+\frac{1}{\widetilde{r}'}-1). \tag{3.2}$$

Indeed, since $(1/\widetilde{r}', 1/r) \in \Delta QRS \setminus ([Q,S] \cup [Q,R])$, it follows that

$$0 \le 1-\frac{n}{2}(\frac{1}{\widetilde{r}'}-\frac{1}{r}) < 1-n(\frac{1}{\widetilde{r}'}-\frac{1}{2}) < 1+\frac{n}{2}(\frac{1}{r}+\frac{1}{\widetilde{r}'}-1) < 1. \tag{3.3}$$

The third inequality in (3.3) says that $(1/\widetilde{r}', 1/r)$ lies above the line joining Q, R. Hence, there exists $1 < \widetilde{q}' < \infty$ such that

$$1-n(\frac{1}{\widetilde{r}'}-\frac{1}{2}) < \frac{1}{\widetilde{q}'} < 1+\frac{n}{2}(\frac{1}{r}+\frac{1}{\widetilde{r}'}-1). \tag{3.4}$$

(Note that the first inequality is also one of the necessary conditions in (1.7).) Now just set $\frac{1}{q} = \frac{1}{\widetilde{q}'} + \frac{n}{2}\left(\frac{1}{\widetilde{r}'}-\frac{1}{r}\right) - 1$. So, (1.5) is satisfied obviously. Then the first inequality in (3.3) gives the second in (3.2), and the first in (3.4) implies the first in (3.2). From (3.2), we can find a small neighborhood V of $(1/\widetilde{r}', 1/r)$, contained in Ω, such that for $(1/a, 1/b) \in V$

$$-\frac{n}{2}(\frac{1}{b}+\frac{1}{a}-1) < \frac{1}{q} \le \frac{1}{\widetilde{q}'} < 1+\frac{n}{2}(\frac{1}{b}+\frac{1}{a}-1).$$

Therefore, by (3.1) we have for $(1/a, 1/b) \in V$

$$\|\delta^{-1} T_\delta F\|_{L^b_x L^q_t} \le C\delta^{\frac{1}{\widetilde{q}'}-\frac{1}{q}+\frac{n}{2}(\frac{1}{a}-\frac{1}{b})-1} \|F\|_{L^a_x L^{\widetilde{q}'}_t}. \tag{3.5}$$

Once this is obtained, we can prove the desired estimates by repeating the argument in the proof of Proposition 2.3. In fact, we consider a point

$(1/a_0, 1/b_0) \in V$ on the line $1/a - 1/b = 1/\widetilde{r}' - 1/r$ and choose two points $(1/a_0, 1/b_i) \in V, i = 1, 2$, such that

$$\frac{1}{\widetilde{q}'} - \frac{1}{q} + \frac{n}{2}(\frac{1}{a_0} - \frac{1}{b_1}) - 1 < 0 < \frac{1}{\widetilde{q}'} - \frac{1}{q} + \frac{n}{2}(\frac{1}{a_0} - \frac{1}{b_2}) - 1.$$

Then replacing (a, b) with (a_0, b_1), (a_0, b_2) in (3.5), we have two estimates to which we can apply Lemma 2.2 with $\varepsilon_i = |\frac{1}{\widetilde{q}'} - \frac{1}{q} + \frac{n}{2}(\frac{1}{a_0} - \frac{1}{b_i}) - 1|$. Hence, we get

$$\Big\| \sum_{\delta \in 2^{\mathbb{Z}}} \delta^{-1} T_\delta F \Big\|_{L_x^{b_0,\infty} L_t^q} \le C \|F\|_{L_x^{a_0} L_t^{\widetilde{q}'}}$$

for all $(1/a_0, 1/b_0) \in V$ if $1/a - 1/b = 1/\widetilde{r}' - 1/r$. We now interpolate these estimates to get the strong type, in particular, at $(1/\widetilde{r}', 1/r)$. This completes the proof.

Proof of Theorem 1.2. First we claim that for $1 \le \widetilde{r}', \widetilde{q}' \le 2 \le r, q \le \infty$, and $0 < \delta \ll 1$,

$$\|T_\delta F\|_{L_x^r L_t^q} \le C \delta^{\frac{1}{\widetilde{r}'} - \frac{1}{r} + \frac{1}{2}(\frac{1}{\widetilde{q}'} - \frac{1}{q})} \|F\|_{L_x^{\widetilde{r}'} L_t^{\widetilde{q}'}} \tag{3.6}$$

whenever $\widehat{F}$ is supported in $\{(\xi, \tau) \in \mathbb{R} \times \mathbb{R} : |\xi| \le 1, |\tau| \sim 1\}$. From (2.3) we see that the Fourier transform of T_δ is essentially supported in the δ-neighborhood of $\{(\xi, \tau) : \tau = -|\xi|^2, |\tau| \sim 1\}$. Hence it is sufficient to show (3.6) by assuming that the Fourier support of F is contained in $\{(\xi, \tau) \in \mathbb{R} \times \mathbb{R} : |\xi| \sim 1, |\tau| \lesssim 1\}$. The contribution from the other region is negligible.

Under this assumption, by recalling (2.3), Plancherel's theorem in t, and Hölder's inequality it follows that

$$\begin{aligned} \|T_\delta F(x, \cdot)\|_{L_t^2} \le & C \Big(\int \Big| \int_{1/2 \le |\xi| \le 2} e^{ix\xi} \, \widehat{\phi}(\frac{\tau + |\xi|^2}{\delta}) \widehat{F}(\xi, \tau) d\xi \Big|^2 d\tau \Big)^{\frac{1}{2}} \\ \le & C \delta^{\frac{1}{2}} \Big(\int \int_{1/2 \le |\xi| \le 2} \big| \widehat{F}(\xi, \tau) \big|^2 d\xi d\tau \Big)^{\frac{1}{2}}. \end{aligned}$$

Plancherel's theorem gives $\|T_\delta F\|_{L_x^\infty L_t^2} \le C \delta^{\frac{1}{2}} \|F\|_{L_x^2 L_t^2}$. By this and duality we have $\|T_\delta F\|_{L_x^\infty L_t^2} \le C \delta \|F\|_{L_x^1 L_t^2}$, and from dispersive estimate $\|T_\delta F\|_{L_x^\infty L_t^\infty} \le C \delta^{\frac{3}{2}} \|F\|_{L_x^1 L_t^1}$. Interpolation between these two estimates gives for $1 \le \widetilde{q}' \le 2 \le q \le \infty$

$$\|T_\delta F\|_{L_x^\infty L_t^q} \le C \delta \delta^{\frac{1}{2}(\frac{1}{\widetilde{q}'} - \frac{1}{q})} \|F\|_{L_x^1 L_t^{\widetilde{q}'}}.$$

Let $Q \subset \mathbb{R}^{1+1}$ be a cube of side length δ^{-1} and $\widetilde{Q}$ be the cube of side length $C\delta^{-1}$ which has the same center as Q. Here $C > 0$ is a sufficiently

large constant. By Hölder's inequality we have for $1 \le \widetilde{r}', \widetilde{q}' \le 2 \le r, q \le \infty$, and $0 < \delta \ll 1$,

$$\|T_\delta(\chi_{\widetilde{Q}} F)\|_{L^r_x L^q_t(Q)} \le C\delta^{\frac{1}{\widetilde{r}'}-\frac{1}{r}+\frac{1}{2}(\frac{1}{\widetilde{q}'}-\frac{1}{q})}\|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}. \tag{3.7}$$

We now deduce (3.6) from this. Firstly, from the assumption that the Fourier transform of F is contained in $\{(\xi,\tau) \in \mathbb{R}\times\mathbb{R} : |\xi| \sim 1\}$, we observe that T_δ is localized at scale δ^{-1} in x. More precisely, the kernel K_δ of T_δ satisfies that

$$|K_\delta(x,t)| \le C\delta^M \chi_{[1/2\delta,\, 2/\delta]}(|t|)(1+|x|)^{-M}$$

for any M if $|x| \ge C\delta^{-1}$. (See (2.5) and the paragraph below it). Hence it follows that if $(x,t) \in Q$, then

$$|T_\delta F(x,t)| \le C|\, T_\delta \chi_{\widetilde{Q}} F(x,t)| + C\delta^M(\mathcal{E}_\delta * |F|)(x,t) \tag{3.8}$$

for some large $M > 0$ where $\mathcal{E}_\delta = \chi_{[1/2\delta,\, 2/\delta]}(t)(1+|x|)^{-M}$. Let $\{Q\}$ be a collection of (essentially disjoint) cubes of side length δ^{-1} which cover $\mathbb{R}^{1+1}$. Then by (3.8) we have

$$\|T_\delta F\|_{L^r_x L^q_t} \le C\Big\|\sum_Q \chi_Q |T_\delta(\chi_{\widetilde{Q}} F)|\Big\|_{L^r_x L^q_t} + C\delta^M\|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}$$

because $r, q \ge \widetilde{r}', \widetilde{q}'$. Hence, by Minkowski's inequality and (3.7) we have

$$\begin{aligned}\|T_\delta F\|_{L^r_x L^q_t} &\le C\Big(\sum_Q \|T_\delta(\chi_{\widetilde{Q}} F)\|^p_{L^r_x L^q_t(Q)}\Big)^{\frac{1}{p}} + C\delta^M\|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}\\ &\le C\delta^{\frac{1}{\widetilde{r}'}-\frac{1}{r}+\frac{1}{2}(\frac{1}{\widetilde{q}'}-\frac{1}{q})}\Big(\sum_Q \|\chi_{\widetilde{Q}} F\|^p_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}\Big)^{\frac{1}{p}} + C\delta^M\|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}\end{aligned}$$

where $p = \min(q,r)$. Since $r, q \ge \widetilde{r}', \widetilde{q}'$, using Minkowski's inequality again, we get the desired inequality (3.6).

For $j \in \mathbb{Z}$, let us define the multiplier operators $\mathcal{P}_j F$ by

$$\widehat{\mathcal{P}_j F}(\xi,\tau) = \phi(2^j\tau)\widehat{F}(\xi,\tau).$$

Using (2.2), (3.6), Lemma 2.2, and repeating the previous argument, one can show that

$$\begin{aligned}\Big\|\mathcal{P}_0\Big(\int_{-\infty}^t e^{i(t-s)\Delta}F(s)ds\Big)\Big\|_{L^r_x L^q_t} &= \Big\|\sum_{\delta\in\mathbb{Z}} \delta^{-1}T_\delta(\mathcal{P}_0 F)\Big\|_{L^r_x L^q_t}\\ &\le C\|\mathcal{P}_0 F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}\end{aligned} \tag{3.9}$$

provided that $1 < \widetilde{r}' < 2 < r < \infty$, $1 \le \widetilde{q}' < 2 < q \le \infty$, and $\frac{1}{\widetilde{r}'} - \frac{1}{r} + \frac{1}{2}(\frac{1}{\widetilde{q}'} - \frac{1}{q}) \ge 1$. In fact, the case $\frac{1}{\widetilde{r}'} - \frac{1}{r} + \frac{1}{2}(\frac{1}{\widetilde{q}'} - \frac{1}{q}) > 1$ can be obtained by direct summation because $\|T_\delta F\|_{L^r_x L^q_t} \le C\|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}$ for $\delta \ge 1$. Now by rescaling it follows that

$$\Big\|\mathcal{P}_j\Big(\int_{-\infty}^t e^{i(t-s)\Delta}F(s)ds\Big)\Big\|_{L^r_x L^q_t} \le C 2^{j(1-\frac{1}{2}(\frac{1}{\widetilde{r}'}-\frac{1}{r})-(\frac{1}{\widetilde{q}'}-\frac{1}{q}))}\|\mathcal{P}_j F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}.$$

Hence we have uniform bounds if $\frac{1}{2}(\frac{1}{\widetilde{r}'} - \frac{1}{r}) + \frac{1}{\widetilde{q}'} - \frac{1}{q} = 1$ and the condition for (3.9) is satisfied. Now note that if $\frac{1}{\widetilde{r}'} - \frac{1}{r} \ge \frac{2}{3}$, there are $\widetilde{q}', q$ satisfying $\frac{1}{2}(\frac{1}{\widetilde{r}'} - \frac{1}{r}) + \frac{1}{\widetilde{q}'} - \frac{1}{q} = 1$ and $\frac{1}{\widetilde{r}'} - \frac{1}{r} + \frac{1}{2}(\frac{1}{\widetilde{q}'} - \frac{1}{q}) \ge 1$. Therefore, if $\frac{1}{\widetilde{r}'} - \frac{1}{r} \ge \frac{2}{3}$ and $1 < \widetilde{r}' < 2 < r < \infty$, we have for some $1 < \widetilde{q}' < 2 < q < \infty$

$$\Big\|\mathcal{P}_j(\int_{-\infty}^t e^{i(t-s)\Delta}F(s)ds)\Big\|_{L^r_x L^q_t} \le C\|\mathcal{P}_j F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}.$$

This can be put together using Littwood-Paley theorem in t. Since $1 < \widetilde{r}' < 2 < r < \infty$ and $1 < \widetilde{q}' \le 2 \le q < \infty$, by Littlewood-paley theorem and Minkowski's inequality

$$\begin{aligned}
\|\sum_j T(\mathcal{P}_j F)\|_{L^r_x L^q_t} &\lesssim \|\big(\sum_j \|T(\mathcal{P}_j F)\|^2_{L^q_t}\big)^{1/2}\|_{L^r_x} \lesssim \big(\sum_j \|T(\mathcal{P}_j F)\|^2_{L^r_x L^q_t}\big)^{1/2} \\
&\lesssim \big(\sum_j \|\mathcal{P}_j F\|^2_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}\big)^{1/2} \lesssim \|\big(\sum_j \|\mathcal{P}_j F\|^2_{L^{\widetilde{q}'}_t}\big)^{1/2}\|_{L^{\widetilde{r}'}_x} \\
&\lesssim \|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}.
\end{aligned}$$

This completes the proof.

4. Necessary conditions

By constructing some counterexamples, we show the conditions (1.8), (1.9).

Proof of (1.9). Let $M > 0$ be a sufficiently large number and let us set

$$\widehat{F}(\xi, \tau) = \varphi(|\xi|)\psi(M^{1/2}(\tau + 1)),$$

where $\psi \in \mathcal{S}(\mathbb{R})$ with supp $\mathcal{F}^{-1}(\psi) \in [0, 1]$ and φ is a smooth function supported in $(1/2, 2)$ with $\varphi(1) = 1$. Note that if $|t| \sim M$, then we may write

$$\int_0^t e^{i(t-s)\Delta}F(s)ds = \int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-it|\xi|^2}\widehat{F}(\xi, -|\xi|^2)d\xi$$

because the support of $F(y, \cdot)$ is contained in $[0, M^{1/2}]$ for all y. Since we have $\int_{S^{n-1}} e^{ix\cdot\xi}d\sigma(\xi) = C|x|^{-\frac{n-2}{2}}J_{\frac{n-2}{2}}(|x|)$, by asymptotic expansion of Bessel

function [21], we see that

$$\left|\int_0^t e^{i(t-s)\Delta}F(s)ds\right| \sim |x|^{-\frac{n-1}{2}}|I(x,t)|$$

for sufficiently large $|x|$, where

$$I(x,t)=\int_0^\infty r^{-\frac{n-1}{2}}\varphi(r)\psi(M^{1/2}(r^2-1))e^{-itr^2}\cos(r|x|-\pi(n-1)/4)dr.$$

We now set $\widetilde{\varphi}=r^{-\frac{n-1}{2}}\varphi(r)$. Then we have

$$\begin{aligned}I(x,t)=&\int \widetilde{\varphi}(1)\psi(M^{1/2}(r^2-1))e^{-itr^2}\cos(r|x|-\pi(n-1)/4)dr\\&+\int(\widetilde{\varphi}(r)-\widetilde{\varphi}(1))\psi(M^{1/2}(r^2-1))e^{-itr^2}\cos(r|x|-\pi(n-1)/4)dr.\end{aligned}$$

By the rapid decay of ψ, the support of $\widetilde{\varphi}$, and the mean value theorem it is easy to see that the second term in the right hand side is $O(M^{-1})$. Similarly, for some large constant $B>0$

$$\begin{aligned}I(x,t)=&\int_{|r-1|\le BM^{-1/2}}\psi(M^{1/2}(r^2-1))e^{-itr^2}\cos(r|x|-\pi(n-1)/4)dr\\&+O(M^{-1/2}/B^{100})+O(M^{-1}).\end{aligned}$$

So we get

$$I(x,t)=\frac{1}{2}\Big(e^{-i\pi(n-1)/4}I_-(x,t)+e^{i\pi(n-1)/4}I_+(x,t)\Big)+O(M^{-1/2}/B^{100}),$$

where

$$I_\pm(x,t)=\int_{|r-1|\le BM^{-1/2}}e^{-i(tr^2\pm r|x|)}\psi(M^{1/2}(r^2-1))dr.$$

By the change of variables $r\to r+1$ and $r\to M^{-1/2}r$, it follows that

$$I_-(x,t)=e^{-i(t-|x|)}M^{-1/2}\int_{|r|\le B}e^{-i\left(tM^{-1}r^2+(2t-|x|)M^{-1/2}r\right)}\psi(2r+M^{-1/2}r^2)dr.$$

So, if $|t|\sim M/B^2$ and $|2t-|x||\lesssim M^{1/2}/B$, we get $|I_-(x,t)|\gtrsim M^{-1/2}$. On the other hand, we see $|I_+(x,t)|\lesssim B^2M^{-1}$ if $|t|\sim M$ and $|x|\sim M$. Consequently, if $|t|\sim M$, $|x|\sim M$ and $|2t-|x||\lesssim M^{1/2}$, then $|I(x,t)|\gtrsim M^{-1/2}$. Therefore, we see that

$$\left\|\int_0^t e^{i(t-s)\Delta}F(s)ds\right\|_{L_x^rL_t^q}\gtrsim M^{-n/2}M^{1/2q}M^{n/r}.$$

Also, it is easy to see that $\|F\|_{L_x^{\tilde{r}'}L_t^{\tilde{q}'}} \lesssim M^{-1/2}M^{1/2\tilde{q}'}$. Hence, the estimate (1.4) implies that

$$M^{-n/2}M^{1/2q}M^{n/r} \lesssim M^{-1/2}M^{1/2\tilde{q}'}.$$

By letting $M \to \infty$, we get the first inequality in (1.9), and the second one follows from duality. $\square$

Proof of (1.8). Let us denote

$$U(F)(x,t) = \int_0^t e^{i(t-s)\Delta}F(s)ds.$$

Then, using the kernel of $e^{it\Delta}$, we see

$$U(F)(x,t) \int_0^t \int_{\mathbb{R}^n} (t-s)^{-n/2} e^{\frac{i|x-y|^2}{4(t-s)}} F(y,s)dyds.$$

For $0 < \delta \ll 1$, we set

$$F(y,s) = \Phi(\delta^{1/2}(y_1+2s), \delta^{1/2}\overline{y}, \delta s)e^{-i(y_1+s)},$$

where $\Phi(y_1,\overline{y},s) = \chi(y_1)\chi(y_2)\cdots\chi(y_n)\chi(s)$ and $\chi = \chi_{[0,1]}$. By the change of variables $y_1 \to y_1 - 2s$, we see that

$$e^{i(x_1+t)}U(F)(x,t) = \int_0^t \int_{\mathbb{R}^n} (t-s)^{-n/2} e^{iP(x,y,t,s)} \Phi(\delta^{1/2}y_1, \delta^{1/2}\overline{y}, \delta s)dyds,$$

where

$$P(x,y,t,s) = \frac{|\overline{x}-\overline{y}|^2 + (x_1-y_1+2t)^2}{4(t-s)}.$$

Note that $|P(x,y,t,s)| \lesssim 1$ if $(x_1+2t)^2 \le \delta^{-1}$, $|\overline{x}| \le \delta^{-1/2}$ and $100\delta^{-1} \le t \le 200\delta^{-1}$. So we see

$$|U(F)(x,t)| \gtrsim \delta^{n/2}\left|\int_0^t \int_{\mathbb{R}^n} \Phi dyds\right| \gtrsim \delta^{-1},$$

provided that $(x_1+2t)^2 \le \delta^{-1}$, $|\overline{x}| \le \delta^{-1/2}$ and $100\delta^{-1} \le t \le 200\delta^{-1}$. Hence

$$\|U(F)\|_{L_x^r L_t^q} \gtrsim \delta^{-1}\delta^{-1/2q}\delta^{-(n+1)/2r}.$$

On the other hand, $\|F\|_{L_x^{\tilde{r}'}L_t^{\tilde{q}'}} \le C\delta^{-1/2\tilde{q}'}\delta^{-(n+1)/2\tilde{r}'}$. From (1.4) we get

$$\delta^{-1}\delta^{-1/2q}\delta^{-(n+1)/2r} \lesssim \delta^{-1/2\tilde{q}'}\delta^{-(n+1)/2\tilde{r}'}.$$

By letting $\delta \to 0$, we get (1.8). $\square$

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